# Math 246C Lecture 5 Notes

Daniel Raban

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## 1 Existence of Lifts, Germs, and Analytic Continuation

### 1.1 Existence of lifts

**Theorem 1.1** (existence of lifts). *Let $X, Y$ be Hausdorff spaces, and let $p : Y \to X$ be a covering map. Let $Z$ be a Riemann surface which is simply connected, and let $f : Z \to X$ be continuous. For any $z_0 \in Z$ and $y_0 \in Y$ such that $f(z_0) = p(y_0)$, there is a unique lift $\tilde{f} : Z \to Y$ such that $\tilde{f}(z_0) = y_0$.*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

*Proof.* Let $z \in Z$, and let $\gamma$ be a path in $Z$ connecting $z_0$ to $z$. Then $\alpha : f \circ \gamma$ is a path in $X$ from $f(z_0)$ to $f(z)$. Let $\tilde{\alpha}$ be the unique lift of $\alpha$ starting with $\tilde{\alpha}(0) = y$. Define $\tilde{f}(z) = \tilde{\alpha}(1)$. This does not depend on the choice of $\gamma$: this follows as $Z$ is simply connected, using the homotopy lifting lemma. Now $p \circ \tilde{f} = f$, so $\tilde{f}$ is a lift of $f$.

We need to check the continuity of $\tilde{f}$. Let $z \in Z$, let $y = \tilde{f}(z)$, and let $V, U$ be neighborhoods of $y, p(y)$, respectively such that $p|_V : V \to U$ is a homeomorphism; $y \in V$ and $f(z) \in U$. $f$ is continuous, so there exists a neighborhood $W$ of $z$ which is path-connected such that $f(W) \subseteq U$. We claim that $\tilde{f}(W) \subseteq W$; this will show the continuity of $\tilde{f}$. Let $z' \in W$, and let $\gamma'$ be a curve in $W$ from $z$ to $z'$. Let $\gamma$ and $\alpha = f \circ \gamma$ be as before. Then $\alpha' = f \circ \gamma' \in U$, so $\tilde{\alpha}'$ sending $t \mapsto (p|_V)(\alpha'(t))$ is a lift of $\alpha'$ starting at $y$. The product curve

$$\tilde{\alpha} * \tilde{\alpha}'(t) = \begin{cases} \tilde{\alpha}(2t) & 0 \leq t \leq 1/2 \\ \tilde{\alpha}'(2t-1) & 1/2 < t \leq 1 \end{cases}$$

is a lift of $\alpha * \alpha' = f(\gamma * \gamma')$. The curve $\gamma * \gamma'$ starts at $z_0$ and ends at $z'$. By definition, $\tilde{f}(z') = \tilde{\alpha} * \tilde{\alpha}'(1) = \tilde{\alpha}'(1) \in V$, where $V$ is a small neighborhood of $y = \tilde{f}(z)$. □

## 1.2 Germs of holomorphic functions

**Definition 1.1.** Let $X$ be a Riemann surface, and let $a \in X$. If $f, g$ are holomorphic near $a$, we say that $f$ and $g$ are **equivalent** if there exists a neighborhood $W$ of $a$ such that $f|_W = g|_W$. The equivalence class of $f$, denoted by $f_a$ is called the **germ** of $f$ at $a$. We let $O_a$ denote the **space of holomorphic germs** at $a$.

**Remark 1.1.** $O_a$ is an algebra (in particular a ring) with no zero divisors.

Let $O_X = \coprod_{a \in X} O_a$. Equip $O_X$ with the following topology. Let $\omega \subseteq X$ be open, and let $f \in \mathrm{Hol}(\omega)$. Set $N(f, \omega) = \{f_x \in O_x : x \in \omega\} \subseteq O_X$. The class of set $N(f, \omega)$ is a base for a topology on $O_X$, where the open sets are all unions of sets of the form $N(f, \omega)$. If $f' \in \mathrm{Hol}(\omega')$, $f'' \in \mathrm{Hol}(\omega'')$, then $N(f', \omega') \cap N(f'', \omega) = N(f', \omega) = N(f'', \omega)$, where $\omega = \{x \in \omega' \cap \omega'' : f'_x = f''_x\}$ is open.

**Definition 1.2.** The topological space $O_X$ is called the **sheaf of germs** of holomorphic functions on $X$.

We have the natural map $p : O_X \to X$ sending $f_a \mapsto a$.

**Proposition 1.1.** *$p$ is a local homeomorphism.*

*Proof.* Let $f_a \in O_X$, and let $(f, \omega)$ be a representative of $f_a$. Then $p : N(f, \omega) \to \omega$ is a homeomorphism. $\square$

**Remark 1.2.** This means that we can given $O_X$ the structure of a Riemann surface. However, this is not a covering map.

**Proposition 1.2.** *The topological space $O_X$ is Hausdorff.*

*Proof.* Let $f_a, g_b \in O_X$ with $f_a \neq g_b$. If $a \neq b$, there exist representatives $(f, \omega_a), (g, \omega_b)$ with $\omega_a \cap \omega_b = \varnothing$ such that $N(f, \omega_a) \cap N(g, \omega_b) = \varnothing$. If $a = b$ and $f_a \neq g_a$, then there exists a connected neighborhood $\omega$ of $a$ and representatives $f(f, \omega), (g, \omega)$ such that $N(f, \omega) \cap N(g, \omega) = \varnothing$ by analytic continuation. $\square$

## 1.3 Analytic continuation

**Definition 1.3.** Let $a \in X$, $f_a \in O_a$, and let $\gamma$ be a curve in $X$ starting at $a$. The **analytic continuation** of $f_a$ along $\gamma$ is a lift $\tilde{\gamma} : [0, 1] \to O_X$ of $\gamma$ such that $\tilde{\gamma}(0) = f_a$.

$$\begin{tikzcd} & O_X \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

We write $\tilde{\gamma}(t) = f_{\gamma(t)} \in O_{\gamma(t)}$.

**Remark 1.3.** The analytic continuation, if it exists, is unique (uniqueness of lifts).

**Example 1.1.** It is not always possible to find an analytic continuation. Let $\gamma(t) = t$ for $0 \leq t \leq 1$, and let $f(z) = 1/(1-z)$ near 0. Then $f$ cannot be analytically continued along the curve $\gamma$.